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Some Harmonic Maps on Pseudo-Riemannian Manifolds

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Rosen's bimetric theory of gravitation, when equipped with a flat background metric, is known to be realized as a harmonic mapping of Minkowski spacetime into a certain homogeneous space. This paper develops and exploits these facts to provide four classes of explicit solutions to Rosen's field equations. These four classes form the elements from which more general solutions may be formed by a type of superposition. It is also shown how spherical gravitational waves may be explicitly built into these solutions.

1. INTRODUCTION

In Stoeger *et al.* (1985) we made a detailed calculation of the curvature of what we called the harmonic mapping space of Rosen's (1974) bimetric theory of gravity, following an earlier model of DeWitt (1967). Our interest in this theory was the fact that the field equations of Rosen were those of a harmonic map between pseudo-Riemannian manifolds, as pointed out earlier by Stoeger (1983). However, our paper exploited neither the harmonic character of the map nor the fact that the harmonic mapping space is a globally affine symmetric space. This paper addresses itself to these aspects, establishing a firm basis for further exploitation of Rosen's theory.

The classical theory of harmonic maps, which is well developed in the context of Riemannian manifolds, does not carry over *in toto* to our situation,

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since our manifolds carry indefinite metrics. For example, there is no maximum principle; hence, local existence/global uniqueness theorems true for harmonic maps on Riemannian manifolds do not carry over to our case. It is necessary to develop the theory of "rank one" harmonic maps begun by Sampson and reworked here, *mutatis mutandis*, in the context of pseudo-Riemannian manifolds.

Our adaptation of Sampson's theorem says that if (M, g) and (N, h) are pseudo-Riemannian manifolds and if $\phi: M \to N$ is a harmonic map having a rank one differential, it can always be factored as $\phi = \sigma \circ \psi$ so that ψ is a real-valued harmonic function, while σ does not have to be totally geodesic, i.e., the image of σ in N does not have to be a geodesic, and the gradient of ϕ does not have to avoid the isotropic subspaces of its domain. If it does, i.e., if its gradient is nonnull, then this forces σ to be a geodesic. This decomposition gives us a means of finding four classes of solutions to Rosen's field equations by applying the well-understood theory of the wave equation to ψ and of geodesics in symmetric spaces to σ. Among these solutions is Rosen's solution. Finally to round out the theory, we give an example due to Lemaire for which σ is not a geodesic.

In order to produce the geodesics in the harmonic mapping space, it is useful to observe, with Misner (1978), that the harmonic mapping space is a homogeneous space, which, when equipped with a certain metric, is a globally affine symmetric space. Here we merely exploit well-known results. However, one point should be mentioned. As many have observed, the harmonic equations do not depend upon the metric in the range space, but only on a connection. In making the curvature calculations in Stoeger *et al.* (1985), we did use a metric (already mentioned above), which we called the DeWitt metric (cf. DeWitt, 1967). Yet this is in some sense superfluous, since the relevant structure is the canonical connection of a globally affine symmetric space.

However, if one wishes to define an energy density of a map and derive the field equations from a variational principle, then a metric becomes essential. Thus, we thought it informative to show which DeWitt metrics are naturally obtained, which are invariant under given transformations of the symmetric space, and which therefore possess the unique canonical symmetric space connection as their Levi-Civita connection.

In Section 2 we develop the relevant facts about rank one harmonic maps. In Section 3 we establish the symmetric space structure of the harmonic mapping space, equipped with any of several compatible DeWitt metrics. We apply these results in Section 4 to obtain four broad classes of solutions to Rosen's field equations. Gravitational waves arise simply because a harmonic function ψ on Minkowski spacetime is in fact a solution to the three-dimensional wave equation.

2. RANK ONE HARMONIC MAPS

We are guided by Eells and Lemaire (1984) for conceptualization and notation. All manifolds and maps are smooth (i.e., C^∞). By metric we mean a pseudo-Riemannian metric, a subclass of which are the Riemannian metrics.

We denote by a pair (M, g) a manifold M with a metric g. The signature of g is given by a pair of nonnegative integers (p, q), where p is the number of negative eigenvalues of g and q its number of positive eigenvalues. We always consider that this metric g induces on M the canonical Levi-Civita connection ${}^g\nabla$.

We are interested in maps $\phi:(M, g)\to(N, h)$ that are harmonic. The differential of ϕ, $d\phi$, is considered a section in the bundle $TM^*\otimes\phi^{-1}TN$, where TM^* is the dual tangent bundle of M and $\phi^{-1}TN$ is the pullback by ϕ of the tangent bundle of N, TN, to the manifold M, i.e., $d\phi$ is a 1-form on M with values in TN. The metrics g and h induce a metric on $TM^*\otimes\phi^{-1}TN$, while the connections ${}^g\nabla$ and ${}^h\nabla$ induce a connection ∇ on $TM^*\otimes\phi^{-1}TN$ that leaves this induced metric invariant. We denote this induced metric of the section $d\phi$ by $\|d\phi\|$. We can consider $\|d\phi\|^2$ as the trace (i.e., contraction) of the $(0, 2)$ covariant tensor ϕ^*h, the pullback of the h metric to M, raised to a $(1, 1)$ mixed tensor by the metric g on M. The quantity $e(\phi)=\frac{1}{2}\|d\phi\|^2$ is the energy density of ϕ. The energy of ϕ is the (extended) real number

$$E(\phi)=\int_M e(\phi)v_g$$

where v_g is the volume element of the metric g. A map ϕ is harmonic if ϕ is an extremal of this energy function, i.e., for any smooth section Y of $\phi^{-1}TN$, the vertical derivative of E at ϕ in the direction Y, $(D_YE)(\phi)$, gives $(D_YE)(\phi)=0$. This is equivalent to satisfying the Euler–Lagrange equation $\tau(\phi)=0$, where $\tau(\phi)$ is the tension field of ϕ, which is a section of $\phi^{-1}TN$, given by $\tau(\phi)=\operatorname{tr}\nabla(d\phi)$. In this expression ∇ is the induced connection on $TM^*\otimes\phi^{-1}TN$, resulting in a section of $TM^*\otimes TM^*\otimes\phi^{-1}TN$, and the trace is taken after raising to a mixed tensor by the metric g.

In local coordinates $\tau(\phi)$ is given as follows. Let (x^i) be local coordinates in M and (y^λ) in N. Let the Christoffel symbols for the connections be ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\lambda_{\mu\nu}$. Then

$$\tau^\lambda(\phi)=g^{ij}\frac{\partial^2\phi^\lambda}{\partial x^i\,\partial x^j}-{}^M\Gamma^k_{ij}\frac{\partial\phi^\lambda}{\partial x^k}+{}^N\Gamma^\lambda_{\mu\nu}\frac{\partial\phi^\mu}{\partial x^i}\frac{\partial\phi^\nu}{\partial x^j}$$

In the applications we make of $E(\phi)$ and $\tau(\phi)$, since M is not necessarily compact, it is possible for $E(\phi)=\pm\infty$. However, since variation is essentially a local phenomenon, variations in E at ϕ by Y are calculated using compactly supported sections, resulting in well-defined values for $(D_Y E)(\phi)$.

Since $d\phi$ is a linear map of fibres of TM to fibres of TN, one can define the concept of *rank* of ϕ at each point of M as the rank of the linear map $d\phi$ between linear spaces. If this rank is constant for all points of M, then we have the rank of ϕ on M.

We are interested in the decomposition of harmonic maps. In this context we need the concept of a totally geodesic map (Eells and Lemaire, 1984, pp. 16–17). A map ϕ: $(M, g) \to (N, h)$ is totally geodesic iff $\nabla d\phi = 0$. This is equivalent to ϕ-preserving connections or to ϕ-preserving geodesics (including the parametrization). Now suppose we have the decomposition

$$\begin{tikzcd}
(M, g) \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & (N, h) \\
& (P, k) \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

If ψ is harmonic and σ is totally geodesic, then ϕ is harmonic.

We exploit this decomposition in the case when ϕ is of rank one. Sampson (1978) studied rank one harmonic maps between Riemannian manifolds. Examining this work and making the modifications always necessary to treat the pseudo-Riemannian case, we state the following:

Theorem 2.1. Let (M, g) and (N, h) be pseudo-Riemannian manifolds with M connected. Suppose $\phi : M \to N$ is harmonic and of rank one on M.

(a) There exists a factorization $\phi = \sigma \circ \psi$ such that ψ is a harmonic function into $\mathbb{R}$.
(b) If $d\psi$ is nonnull, then the image of σ is a geodesic of N.

Proof. (a) The rank one hypothesis on ϕ allows us to affirm the following. Locally we have coordinate charts U of M with coordinates (x^i) and coordinate charts V of N with coordinates (y^α) such that $\phi(V) \subset V$ and $\phi|_U$ has its image mapping onto the first coordinate $y^1 = \phi^1$ ($\phi^\alpha = 0$, $\alpha \neq 1$). Taking this coordinate as the open interval I_U, this determines a surjective map $\psi_v : U \to I_v$ and a bijection $\sigma_v : I_v \to \phi(U)$, giving the factorization $\phi|_U = \sigma_U \circ \psi_U$.

In these coordinates the harmonic equations for ϕ become

$$0 = g^{ij}\left(\frac{\partial^2 \phi^1}{\partial x^i\, \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial \phi^1}{\partial x^k} + {}^N\Gamma^1_{11}\frac{\partial \phi^1}{\partial x^i}\frac{\partial \phi^1}{\partial x^j}\right) \tag{1}$$

$$0 = g^{ij}\, {}^N\Gamma^\lambda_{11}\frac{\partial \phi^1}{\partial x^i}\frac{\partial \phi^1}{\partial x^j}, \qquad \lambda \neq 1$$

Since $\phi^1 = \psi_U$, the first equation becomes

$$0 = g^{ij}\left(\frac{\partial^2 \psi_U}{\partial x^i\,\partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial \psi_U}{\partial x^k} + {}^N\Gamma^1_{11}\frac{\partial \psi_U}{\partial x^i}\frac{\partial \psi_U}{\partial x^j}\right)$$

Now changing only the first coordinate in V, we seek

$$(y, y^2, \ldots, y^n) \to (z(y), y^2), \ldots, y^n)$$

where $z(y)$ is a function of the first coordinate only. Letting ${}^N\bar{\Gamma}^\lambda_{\mu\nu}$ be the Chirstoffel symbols in the new coordinates, we know

$${}^N\Gamma^1_{11} = \left(\frac{\partial z}{\partial y}\right)^{-1}\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\,{}^N\bar{\Gamma}^1_{11}$$

This results in an ordinary differential equation, which for given initial data has a unique solution for z such that ${}^N\bar{\Gamma}^1_{11} = 0$. Thus, reparametrizing the interval I_U by z, we obtain the equation

$$0 = g^{ij}\left(\frac{\partial^2 \psi_U}{\partial x^i\,\partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial \psi_U}{\partial x^k}\right)$$

If I_U is now given its canonical connection, this says that ψ_U is a harmonic function. Uniqueness and connectedness then gives the global factorization $\phi = \sigma \circ \psi$, with ψ a harmonic function into $\mathbb{R}$.

(b) In order that $\sigma_U(I_U) = \phi(U)$ (with parameter z) be a geodesic, it is sufficient that ${}^N\bar{\Gamma}^\lambda_{11} = 0$, $\lambda \neq 1$ (which in our case is equivalent to ${}^N\Gamma^\lambda_{11} = 0$, $\lambda \neq 1$). But equations (1) are equivalent to

$$0 = g^{ij}\,{}^N\Gamma^\lambda_{11}\frac{\partial \psi_U}{\partial x^i}\frac{\partial \psi_U}{\partial x^j}, \qquad \lambda \neq 1$$

Thus, if $d\psi$ is nonnull, this gives ${}^N\Gamma^\lambda_{11} = 0$, $\lambda \neq 1$. We conclude that $\sigma_U(I_U) = \phi(U)$ is a geodesic in N. ■

We observe the following. For rank one harmonic maps ϕ, in the Riemannian case it is always true that $\phi = \sigma \circ \psi$ with ψ harmonic and the image of σ a geodesic in N. Thus, the sufficient conditions for a composition of maps to be harmonic are also in this case necessary conditions. However, in the pseudo-Riemannian case this is not true. We can always conclude that ψ is harmonic (and this independent of the null condition on $d\psi$), but that the image of σ need not be a geodesic in N. L. Lemaire (personal communication) gives the following example. Let M and N both be $\mathbb{R}^2$ with Minkowski metric. Define $\phi\colon M \to N$ by

$$\phi(x, y) = (u, v) = (\exp(x - y), \exp^2(x - y))$$

Now ϕ is harmonic and of rank one. In this case $\psi:\mathbb{R}^2\to(0,\infty)$ is $\psi(x,y)=s=\exp(x-y)$. Now ψ is harmonic and $d\psi$ is null. The map $\sigma:(0,\infty)\to\mathbb{R}^2$ is $\sigma(s)=(u,v)=(s,s^2)$, which is *not* a geodesic in $\mathbb{R}^2$ with Minkowski metric. Indeed, with this ψ and *any* σ, $\phi=\sigma\circ\psi$ is harmonic. This phenomenon generalizes into the following:

Corollary 2.2. Let (M,g) and (N,h) be pseudo-Riemannian manifolds with M connected. Suppose $\phi:M\to N$ is of rank one on M. Let ϕ have the factorization $\phi=\sigma\circ\psi$. If ψ is harmonic and $d\psi$ is null, then ψ is harmonic.

Proof. As in the proof of Theorem 2.1, rank one allows us to write the tension of $\phi|_U=\sigma_U\circ\psi_U$ as

$$\tau^1(\phi|_U)=g^{ij}\left(\frac{\partial^2\phi^1}{\partial x^i\partial x^j}-{}^M\Gamma^k_{ij}\frac{\partial\phi^1}{\partial x^k}\right)$$

$$\tau^\lambda(\phi|_U)=g^{ij}\,{}^N\Gamma^\lambda_{11}\frac{\partial\phi^1}{\partial x^i}\frac{\partial\phi^1}{\partial x^j},\qquad \lambda\neq 1$$

Since $\phi^1=\psi_U$, the harmonic hypothesis on ψ makes $\tau^1(\phi|_U)=0$, and the null condition on $d\psi$ makes $\tau^\lambda(\phi|_U)=0$, $\lambda\neq 1$. ■

We also make some comments about the energy density $e(\phi)$. In our local coordinates of Theorem 2.1, where the parameter for σ_U is s, we have

$$2e(\phi|_U)=2(\sigma_U\circ\psi_U)=\left(g^{ij}\frac{\partial\psi_U}{\partial x^i}\frac{\partial\psi_U}{\partial x^j}\right)\times\left((h_{\mu\nu}\circ\sigma_U\circ\psi_U)\left(\frac{d\sigma^\mu}{ds}\circ\psi_U\right)\left(\frac{d\sigma^\nu}{ds}\circ\psi_U\right)\right)$$

If $d\psi=0$, then $e(\phi)=0$. If $d\psi\neq 0$, then $e(\phi)=0$ means the image of ϕ, which is a geodesic in N by Theorem 2.1, is an isotropic geodesic in N. Finally, if $e(\phi)\neq 0$, then $d\psi$ is not null and the image of ϕ is a nonisotropic geodesic in N.

3. THE SYMMETRIC SPACE STRUCTURE OF $S(p,q)$ AND METRICS ON $S(p,q)$

Our interest in this paper is to give rank one solutions to Rosen's equations for his theory of gravitation. [See Stoeger *et al.* (1985) for a discussion of this theory.) His vacuum equations (homogeneous equations) are harmonic maps from $\mathbb{R}^4$ with a Minkowski metric to $S(1,3)$, the set of all symmetric, nondegenerate $(0,2)$ tensors with signature $(1,3)$ (1 negative, 3 positive). Thus, following the conclusions of Section 2, we need to expose the geodesic structure of $S(p, q)$, which now will mean the set of all symmetric, nonsingular matrices of signature (p, q) (p negative, q positive) in $GL(m, \mathbb{R})$, $m = p + q$. We merely summarize standard well-known results in this section.

If one analyzes the natural representation of $GL(m, \mathbb{R})$ on the $(0, 2)$ tensors of $\mathbb{R}^m$, one finds the following action:

$$GL(m, \mathbb{R}) \times S(p, q) \to S(p, q)$$
$$(k, g) \mapsto kgk^T$$

where k^T is the transpose of k. For reasons which appear later, we will call this natural action the *squaring action* on $S(p, q)$. We note the following: $GL(m, \mathbb{R})$ has two components, whose identity component is $GL^+(m, \mathbb{R})$. In our applications we cannot restrict this action to this connected component. As a consequence, in general we do not have an effective action even when m is odd, since I_m and $-I_m$ both give the identity transformation. The action is transitive, and the isotropy subgroup of the "canonical" matrix in $S(p, q)$,

$$I_{p,q} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix}$$

is by definition the pseudoorthogonal group $O(p, q)$. Thus, we have the following diffeomorphism:

$$GL(m, \mathbb{R})/O(p \cdot q) \to S(p, q) \tag{2}$$
$$[k] \mapsto kI_{p,q}k^T$$

where $[k]$ is the coset containing k. The coset action of $GL(m, \mathbb{R})$ goes over to the squaring action on $S(p, q)$. We note that $S(p, q)$ is connected.

There is a natural symmetry $\sigma_{p,q}$ on $GL(m, \mathbb{R})$, i.e., $\sigma_{p,q}$ is a diffeomorphism of $GL(m, \mathbb{R})$ such that $\sigma_{p,q}^2 = \text{identity}$:

$$\sigma_{p,q}: GL(m, \mathbb{R}) \to GL(m, \mathbb{R})$$
$$k \mapsto \sigma_{p,q}(k) = I_{p,q}k^{T-1}I_{p,q}$$

where k^{T-1} denotes the transpose inverse of k. Its fixed point set is $O(p, q)$. Thus, there is an induced symmetry on $GL(m, \mathbb{R})/O(p, q)$ which maps cosets $[k]$ into cosets $[\sigma_{p,q}(k)]$. Using the diffeomorphism (2), we have a symmetry defined on $S(p, q)$:

$$s_O: S(p, q) \to S(p, q)$$
$$g \mapsto s_O(g) = I_{p,q}g^{-1}I_{p,q}$$

with an isolated fixed point $I_{p,q}$.

Let $gl(m, \mathbb{R})$ and $so(p, q)$ be the Lie algebras of $GL(m, \mathbb{R})$ and $O(p, q)$, respectively, considered, if necessary, as sets of tangent vectors at the identity I_m of $GL(m, \mathbb{R})$. We have a decomposition of $gl(m, \mathbb{R})$ as a direct sum of linear spaces

$$gl(m, \mathbb{R}) = so(p, q) \oplus \omega$$

The Lie algebra $so(p, q)$ consists of all matrices of the form

$$A = \begin{bmatrix} A_1 & A_2^T \\ A_2 & A_4 \end{bmatrix}$$

where A is blocked by p and q; A_1 and A_4 are skew symmetric; and A_2 is arbitrary. The linear subspace ω consists of all matrices of the form

$$Z = \begin{bmatrix} Z_1 & -Z_2^T \\ Z_2 & Z_4 \end{bmatrix}$$

where Z is also blocked by p and q; Z_1 and Z_4 are symmetric; and Z_2 is arbitrary. Thus we have

$$\mathrm{Ad}(O(p, q))\omega \subset \omega$$

$$[so(p, q), \omega] \subset \omega$$

The differential of $\sigma_{p,q}$ decomposes the tangent space at the identity also into $gl(m, \mathbb{R}) = so(p, q) \oplus \omega$, where $so(p, q)$ is the $+1$ eigenspace and ω the -1 eigenspace. Thus,

$$[\omega, \omega] \subset so(p, q)$$

This means that $GL(m, \mathbb{R})/O(p, q)$ is a reductive homogeneous space in the sense of Kobayashi and Nomizu (1969, Vol. II, p. 190). By Kobayashi and Nomizu (1969, Vol. II, p. 192)] there exists a canonical connection on $GL(m, \mathbb{R})/O(p, q)$ which is invariant by $GL(m, \mathbb{R})$ by coset action, i.e., $GL(M, \mathscr{R})$ is a group of affine transformations of this connection. Because of the symmetry condition, the symmetry is also an affine transformation, and thus $GL(m, \mathbb{R})/O(p, q)$ with the canonical connection is a globally affine symmetric space, i.e., the torsion $T = 0$ and the covariant differential of the curvature $\nabla R = 0$. This connection is also complete.

We note here that these results were explicitly calculated in Stoeger *et al.* (1985), where in addition the actual curvatures were determined.

We now transfer this structure to $S(p, q)$ by (2). Thus $S(p, q)$ becomes a complete, globally affine symmetric space in which the symmetry and the elements of the group $GL(m, \mathbb{R})$ acting by the squaring action are affine transformations.

We wish to identify the geodesics in $S(p, q)$. In $GL(m,\mathbb{R})/O(p,q)$, from Kobayashi and Nomizu (1969, Vol. II, p. 231) we know the geodesics are of the form $\exp(tZ)\cdot[O(p,q)]$ for $Z\in\omega$. In $S(p,q)$ this translates to $\exp(tZ)I_{p,q}\exp(tZ)^T$ for geodesics starting at $I_{p,q}$ in the direction $ZI_{p,q}+I_{p,q}Z^T$.

We now change perspective and consider $S(p,q)\subset GL(m,\mathbb{R})$. Since $I_{p,q}Z^T=ZI_{p,q}$, the geodesic reduces to $\exp(2tZ)I_{p,q}$ and the direction reduces to $2ZI_{p,q}$. (The appearance of the factor 2 in these expressions is the motive for naming the *squaring action* as was done above.) We can conclude that one-parameter subgroups in $GL(m,\mathbb{R})$, $\exp(tz^T)$, starting at I_m in the direction $Z^T\in\omega$, left-translated by $I_{p,q}$, give geodesics in $S(p,q)$ starting at $I_{p,q}$ in the direction $I_{p,q}Z^T$. Thus, we have reduced the study of the globally affine symmetric space $S(p,q)$ to how $S(p,q)$ sits in $GL(m,\mathbb{R})$.

We gather these results in the following:

Geodesic Proposition. The geodesic σ in $S(p,q)$ through $I_{p,q}$, with initial tangent X an arbitrary symmetric matrix, is

$$\sigma(t)=I_{p,q}\exp(tI_{p,q}X)$$

In particular, we exploit the decomposition of $GL(m,\mathbb{R})=SL^{\pm}(m,\mathbb{R})\cdot\mathbb{R}^{+}I_m$, where the elements of $SL^{\pm}(m,\mathbb{R})$ are all $(m\times m)$ matrices of $\det=\pm1$ and $\mathbb{R}^{+}$ is the set of the positive real numbers. We let $SS(p,q)=SL^{\pm}(m,\mathbb{R})\cap S(p,q)$. We have $S(p,q)=SS(p,q)\cdot\mathbb{R}^{+}I_m$, and $SS(p,q)$ is also a globally affine symmetric space by the same arguments as above. Considering $SS(p,q)\subset SL^{\pm}(m,\mathbb{R})$, we identify one-parameter subgroups in $SL^{\pm}(m,\mathbb{R})$, $\exp(tZ)$, starting at I_m in the direction $Z\in\tilde{\omega}$, where $\tilde{\omega}$ is the set of trace $=0$ matrices of ω, and the Lie algebra of $SL^{\pm}(m,\mathbb{R})$, $sl(m,\mathbb{R})=so(p,q)\oplus\tilde{\omega}$. These one-parameter subgroups are left-translated by $I_{p,q}$ to give geodesics in $SS(p,q)$ that are thus geodesics in $S(p,q)$ also. In other words, $SS(p,q)$ is a totally geodesic submanifold of $S(p,q)$, as are all translates $SS(p,q)\cdot\lambda I_m$, $\lambda\in\mathbb{R}^{+}$ (cf. Kobayashi and Nomizu, 1969, Chapter XI.4).

We remark that if $S(p,q)$ is given a pseudo-Riemannian metric G such that the squaring action of $GL(m,\mathbb{R})$ is also an isometry with respect to G, then the Levi-Civita connection ${}^{G}\nabla$ on $S(p,q)$ coincides with the canonical connection on $S(p,q)$ as a globally affine symmetric space (cf. Kobayashi and Nomizu, 1969, Vol. II, p. 232). This allows us to derive these harmonic maps from a variational principle on a Lagrangian. We exhibit a set of metrics on $S(p,q)$. Any metric g on M naturally induces a metric on $T^0_2(TM)$ by

$$G(\omega^1\otimes\eta^1,\omega^2\otimes\eta^2)=g(\omega^1,\omega^2)g(\eta^1,\eta^2)$$

Using a coordinate chart (x^α) on M, this determines the coordinates to which the matrices $S(p, q)$ were referenced, i.e., coordinates $(g_{\alpha\beta})$ determined by $g=\sum_{\alpha\beta} g_{\alpha\beta} dx^\alpha \otimes dx^\beta$. Now the natural isomorphism between $T(S(p, q))_g$ and $S_2^0(\mathbb{R}^m)$ given by this coordinate system translates this induced metric to a metric in $S(p, q)$ expressed in coordinates as

$$G = \sum_{\alpha\beta\lambda\rho} g^{\alpha\lambda} g^{\beta\rho} dg_{\alpha\beta} \otimes dg_{\lambda\rho}$$

This can be expressed by a trace form. For $X,\ Y \in T(S(p, q))_g$,

$$G_g(X, Y) = \operatorname{tr}(g^{-1} X g^{-1} Y)$$

Note that $g^{-1}X$ and $g^{-1}Y$ belong to ω.

We now exploit the fact $S(p, q) = SS(p, q) \cdot \mathbb{R}^+ I_m$. We choose the following explicit diffeomorphism:

$$S(p, q) \to SS(p, q) \times \mathbb{R}^+$$
$$g \to (\bar{g}, \lambda)$$

where $\bar{g} = |g|^{-1/m} g$, $\lambda = |g|^{1/m}$, and $|g| = |\det g|$. Introducing a product metric G on $SS(p, q) \times \mathbb{R}^+$, we have for $g = \lambda \bar{g}$, $X = \lambda \bar{X} + \rho_1 \bar{g}$, and $Y = \lambda \bar{Y}^+ \rho_2 \bar{g}$,

$$G_g(\lambda \bar{X} + \rho_1 \bar{g}, \lambda \bar{Y} + \rho_2 \bar{g})$$
$$= \operatorname{tr}(\bar{g}^{-1} \bar{X}_{\bar{g}}^{-1} \bar{Y}) + \frac{1}{m} \operatorname{tr}(\lambda^{-1} \rho_1 I_m) \operatorname{tr}(\lambda^{-1} \rho_2 I_m)$$

This depends on the fact that $\bar{g}^{-1}\bar{X},\ \bar{g}^{-1}\bar{Y} \in \bar{\omega}$, i.e., $\operatorname{tr}(\bar{g}^{-1}\bar{X}) = \operatorname{tr}(\bar{g}^{-1}\bar{Y}) = 0$.

This motivates a natural extension of this metric to an arbitrary metric on the $\mathbb{R}^+$ part. For $X,\ Y \in T(S(p, q))_g$,

$$G_g(X, Y) = a \operatorname{tr}(g^{-1} X g^{-1} Y) + b \operatorname{tr}(g^{-1} X) \operatorname{tr}(g^{-1} Y)$$

for $\mu,\ \eta \in T(S(p, q))_g^*$ in the cotangent bundle of $S(p, q)$,

$$G_g(\mu, \eta) = c \operatorname{tr}(g_\mu g_\eta) + d \operatorname{tr}(g_\mu) \operatorname{tr}(g_\eta)$$

with

$$c = a^{-1}, \qquad a \neq 0, \qquad d = -\frac{b}{a} \frac{1}{a+mb}, \qquad \frac{b}{a} \neq -\frac{1}{m}$$

This effectively merely adjusts the metric on the $\mathbb{R}^+$ part, since

$$G_g(X, Y) = a \operatorname{tr}(\bar{g}^{-1} \bar{X} \bar{g}^{-1} \bar{Y}) + \frac{a+mb}{m} \operatorname{tr}(\lambda^{-1} \rho_1 I_m) \operatorname{tr}(\lambda^{-1} \rho_2 I_m)$$

What is important is that these metrics on $S(p, q)$ are invariant by the symmetry and by the squaring action. Thus, the Levi-Civita connection that each one defines coincides with the unique canonical connection on $S(p, q)$ as a symmetric space (cf. Kobayashi and Nomizu, 1969, Vol. p. 232).

We indicate the signature of these metrics. It is sufficient to seek an orthogonal basis for $T(S(p,q))_{I_{p,q}}$. A basis for the $\mathbb{R}^+$ part is $I_{p,q}$ and

$$G_{I_{p,q}}(I_{p,q}, I_{p,q}) = m(a+mb)$$

Thus, depending on the choices made for a and b, the contribution of the $\mathbb{R}^+$ part to the signature could be either one positive or one negative. It is easy to construct a basis of diagonal matrices for the $SS(p,q)$. They contribute to the signature either

$$(pq, \tfrac{1}{2}(m)(m+1)-pq-1), \qquad a>0$$

$$(\tfrac{1}{2}(m)(m+1)-pq-1, pq), \qquad a<0$$

In particular, for $S(1,3)$ with $a=1$ and $b=-1$, the signature on the $\mathbb{R}^+$ part is one negative, while on the $SS(1,3)$ part, the signature is $(3,6)$.

4. FOUR CLASSES OF SOLUTIONS TO ROSEN'S FIELD EQUATIONS

In this section we apply the results of Sections 2 and 3. Let us first relate them to Rosen's static, spherically symmetric solution to his field equations, namely

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

where $g=(g_{ij})$ is the matrix

$$g = \operatorname{diag}(-\exp(-2m_1/r), \exp(2m_2/r),$$
$$\exp(2m_2/r), \exp(2m_2/r))$$

Here r is the spatial distance (from the origin) of the point at which g is evaluated. This solution is rank one, since the matrix function g factors into the composition of the standing wave

$$\psi = 1/r$$

with the geodesic in $S(1,3)$

$$\sigma(s) = \operatorname{diag}(-\exp(-sm_1), \exp(sm_2), \exp(sm_2), \exp(sm_2))$$

Here m_1 and m_2 represent two masses. More abstractly, they are constants of integration. In the notation of the geodesic proposition, one expresses σ as

$$\sigma(s) = I_{1,3}\exp(sI_{1,3}X) \tag{3}$$

where

$$X = \operatorname{diag}(m_1, m_2, m_2, m_2)$$

A slightly more general form which could be used would be

$$X = \mathrm{diag}(m_1, m_2, m_3, m_4)$$

However, the spatial asymmetry in the use of four masses would correspond to an aspherical gravitational field.

One has three other classes of geodesics that can be written down explicitly. We will use dt for dx^0, dx for dx^1, dy for dx^2, and dz for dx^3 with which to express the line elements of the corresponding Lorentzian metrics that are the solutions of Rosen's field equations. Thus, for example, dt^2 means $(dx^0)^2$.

We first write down the line element of the metric and then the corresponding value of $I_{1,3}X$ to be used in equation (3). The value of ψ is that of the standing wave above, $\psi = 1/r$. Later we shall replace ψ with other solutions to the three-dimensional wave equation in order to introduce even horizons.

Line element:

$$\text{(i)} \quad ds^2 = -(1-a/r)dt^2 + (1+a/r)dx^2 + (dy^2 + dz^2)$$

Corresponding geodesic tangent vector in $S(1,3)$ (blank off-diagonal entries denote zeros):

$$\text{(Xi)} \quad I_{1,3}X = \begin{vmatrix} -a & -a & & \\ a & a & & \\ & & 0 & \\ & & & 0 \end{vmatrix}.$$

Line element:

$$\begin{aligned} \text{(ii)} \quad ds^2 = {} & [-\cos(\mu/r) + a/\mu \sin(\mu/r)]\, dt^2 \\ & + [2b/\mu \sin(\mu/r)]\, dt\, dx \\ & + [\cos(\mu/r) + a/\mu \sin(\mu/r)]\, dx^2 \\ & + (dy^2 + dz^2) \end{aligned}$$

Geodesic tangent vector:

$$\text{(Xii)} \quad I_{1,3}X = \begin{vmatrix} -a & -b & & \\ b & a & & \\ & & 0 & \\ & & & 0 \end{vmatrix}$$

where $b^2 > a^2$ and $\mu = (b^2 - a^2)^{1/2}$.

Line element:

$$\text{(iii)}\quad ds^2=[-\cosh(\mu/r)+a/\mu\,\sinh(\mu/r)]\,dt^2+[2b/\mu\,\sinh(\mu/r)]\,dt\,dx+[\cosh(\mu/r)+a/\mu\,\sinh(\mu/r)]\,dx^2+(dy^2+dz^2)$$

Geodesic tangent vector:

$$\text{(Xiii)}\quad I_{1,3}X=\begin{vmatrix} -a & -b & & \\ b & a & & \\ & & 0 & \\ & & & 0 \end{vmatrix}$$

where $a^2>b^2$ and $\mu=(a^2-b^2)^{1/2}$.

There is one other class, that for which the line element is

$$\text{(iv)}\quad ds^2=-\exp(m/r)\,dt^2+\exp(m/r)\,dx^2+\exp(p/r)\,dy^2+\exp(q/r)\,dz^2$$

Geodesic tangent vector:

$$\text{(Xiv)}\quad I_{1,3}X=\begin{vmatrix} m & & & \\ & m & & \\ & & p & \\ & & & q \end{vmatrix}$$

Since a matrix of the form (Xiv) commutes with matrices of each of the previous three forms, one may add it to each, the resulting exponential matrices being the product of each of the exponentials of the component matrices. For example, adding a matrix of the form of (Xiv) to one of the form (Xi) and exponentiating the result (and as usual composing with the standing wave $\psi=1/r$) yields a solution to Rosen's field equations of the form

$$\text{(v)}\quad ds^2=\exp(m/r)[(-1+a/r)\,dt^2+(1+a/r)\,dx^2]+\exp(p/r)\,dy^2+\exp(q/r)\,dz^2$$

To avoid the evident asymmetry in this solution, one must take $m=p=q$. Notice that with $m=p=q$, this would be a static axially symmetric solution without any angular momentum, but with an event horizon at $r=a$, where the speed of light falls to zero.

The most general form that $I_{1,3}X$ can take, after rotating space coordinates, would be

$$\text{(Xvi)} \quad I_{1,3}X = \begin{vmatrix} a_1 & -b & & \\ b & a_2 & c & \\ & c & p & n \\ & & n & q \end{vmatrix}$$

It is easy to see that if two out of the three of b, c, or n is zero or c is zero, then this latter $I_{1,3}X$ by a further rotation of space coordinates may be put into one of the three forms: (Xi)+(Xiv), (Xii)+(Xiv), or (Xiii)+(Xiv). However, unless two out of the three of b, c, or n is zero or c is zero, it is not clear what the exponential of $I_{1,3}X$ is.

By replacing $\psi = 1/r$ in each of the above line elements by a nonstationary wave, one can obtain solutions to Rosen's field equations that admit gravitational waves. By replacing $\psi = 1/r$ in Rosen's static, spherically symmetric solution with a time-dependent ψ, one can obtain explicit spherically symmetric black holes that generate gravitational waves. It is not hard to see how to provide solutions ψ to the three-dimensional wave equation that yield event horizons that expand or collpase.

To preserve spherical symmetry, it is necessary to give the solution ψ to the three-dimensional wave equation in the form

$$\psi(t, x, y, z) = f(t, r)/r$$

where $f(t, u)$ is a solution to the one-dimensional wave equation and r is the Euclidean norm of the space vector (x, y, z). For example, if we take for $f(t, u)$ the function

$$f(t, u) = 1/(u-a-t) + 1/(u-a+t)$$

then the Rosen static, spherically symmetric solution becomes the time-dependent solution

$$\begin{aligned} ds^2 &= \exp\{-m_1(1-a/r)/[(r-a)^2 - t^2]\, dt^2 \\ &\quad + \exp\{m_2(1-a/r)/[(r-a)^2 - t^2]\}(dx^2 + dy^2 + dz^2) \end{aligned}$$

which has an expanding event horizon at $r - a = t$.

REFERENCES

DeWitt, B. S. (1967). Quantum theory of gravity I. The canonical theory, *Physical Review*, **160**, 1113–1148.

Eells, J., and Lemaire, L. (1984). Selected Topics in Harmonic Maps, CBMS Regional Conference Lecture Notes No. 50.

Helgason, S. (1978). *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York.

Kobayashi S., and Nomizu K. (1969). *Foundations of Differential Geometry*, Interscience, New York.

Misner, C. W. (1978). Harmonic maps as models for physical theories, *Physical Review D*, **18**, 4510-4524.

Rosen, N. (1974). A theory of gravitation, *Annals of Physics*, **84**, 455-473.

Sampson, J. H. (1978). Some properties and applications of harmonic mappings, *Annales Scientifiques de l'Ecole Normale Superieur*, **11**, 211-228.

Stoeger, W. R. (1983). Rosen's bi-metric theory of gravity as a harmonic map, in *Proceedings of the Third Marcel Grossman Meeting on Recent Developments in General Relativity*, Hu Ning, ed., North-Holland, Amsterdam, Part B, pp. 921-925.

Stoeger, W. R., Whitman, A. P., and Knill, R. J. (1985). The harmonic mapping character of Rosen's bi-metric theory of gravity and the geometry of its harmonic mapping space, *Journal of Mathematical Physics*, **26**, 2032-2038.